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Quantum Lie Algebras of Type A_n

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Abstract

It is shown that the quantised enveloping algebra of $\mathfrak{sl}(n)$ contains a quantum Lie algebra, defined by means of axioms similar to Woronowicz's. This gives rise to Lie algebra-like generators and relations for the locally finite part of the quantised enveloping algebra, and suggests a canonical Poincaré-Birkhoff-Witt basis.

1 Introduction

It is generally believed that quantised enveloping algebras are not enveloping algebras; Chari and Pressley, for example, assert [3, p. 258] that “there is no quantum Lie algebra” underlying a quantised enveloping algebra $U_q(\mathfrak{g})$. The purpose of this paper is to challenge this belief. The ultimate intention is to describe all quantised enveloping algebras as genuine enveloping algebras — that is, as associative algebras generated by a finite-dimensional system like a Lie algebra, with relations obtained by interpreting the Lie brackets as quadratic expressions like commutators. Here we succeed, for the case where

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\mathfrak{g} is of type A_n , in finding generators and relations which are essentially of the desired form; moreover, they appear to yield a Poincaré-Birkhoff-Witt theorem for the enveloping algebra. They are based on Lie brackets defined by means of the adjoint action of a Hopf algebra on itself, the quantum Lie algebra being a certain ad-invariant subspace of the quantised enveloping algebra. A generalisation of the classical concept of Lie algebra emerges, which is close to that found by Woronowicz [14] in his theory of non-commutative differential geometry on quantum groups. However, Woronowicz's theory yields a satisfactory deformation of the classical Lie algebra only in the case of general linear groups [12, 11, 9]; in all other cases (see [7, 13, 2, 10]) the quantum Lie algebra has a different dimension from the corresponding classical one: it is always n^2 where n is the dimension of a particular representation. In particular, quantum Lie algebras corresponding to $\mathfrak{sl}(n)$ and having dimension $n^2 - 1$, such as are described in this paper, appear to be new.

The following reservations must be made:

1. The relations for the quantised enveloping algebra are not, as expected, quadratic-linear (deformed commutator = Lie algebra element) but homogeneous quadratic, the right-hand side containing an extra central element as a factor. To put it another way, the structure constants are multiples of a function of the Casimirs. This is an interesting feature; it means that the PBW theorem leads to a description of the vector space structure of the enveloping algebra as a space of polynomial functions not on a flat space (namely the Lie algebra) but on a hypersurface of the same dimension. Thus quantisation gives rise to curvature of the enveloping algebra.

2. The algebra generated by our quantum Lie algebra cannot be expected to be the same size as the quantised enveloping algebra, since this differs from the classical enveloping algebra in containing infinite power series (specifically, exponentials) in the generators. On the other hand, if, as is common in the mathematical literature, one takes the generators of the quantised enveloping algebra to be q^{H_i} in place of the Cartan subalgebra elements H_i , then one loses the polynomials in the H_i . It is a remarkable fact [6] that the quantised enveloping algebra so defined contains a subalgebra, its locally finite part, which has the same structure as the classical enveloping algebra and is a large part of the quantised enveloping algebra. It seems likely that the enveloping algebra of the quantum Lie algebra, as defined in this paper, coincides with the locally finite part of the quantised enveloping algebra. In this paper this is proved for the case of $\mathfrak{sl}(2)$.

3. The construction of a quantum Lie algebra given here makes sense for any simple Lie algebra, but it is only for the Lie algebras of type A_n that it yields a system with the same dimension as the classical Lie algebra. Thus it

is only in these cases that a PBW theorem is likely to hold. Even here, I have no general proof of the theorem; it can be checked by explicit calculation for A_1 and A_2 .

4. As indicated above, the commutator-like relations obtained from the quantum Lie algebra are not the only relations in the quantised enveloping algebra; there is also a relation between central elements, giving the extra (non-classical) central generator in terms of more familiar Casimir elements which are polynomials in Lie algebra elements. The existence of this extra relation is at present only a conjecture in all cases except $\mathfrak{sl}(2)$, for which it can be explicitly demonstrated.

The organisation of the paper is as follows. The next section contains the definition of a quantum Lie algebra and the proof that such a system can be found inside the quantised enveloping algebra. This relies heavily on the theorems of Joseph and Letzter [6] concerning the locally finite part of a quantised enveloping algebra. In section 3 we complete the theory for $\mathfrak{sl}(2)_q$, proving the Poincaré-Birkhoff-Witt theorem, determining the extra relation between central elements, and proving that the algebra defined by the quantum Lie algebra relations, together with the extra relation, is the locally finite part of the quantised enveloping algebra. Section 4 contains some material on the quantum Lie algebra $\mathfrak{sl}(3)_q$, with formulae for the Lie brackets and a discussion of the rather surprising symmetry properties of the enveloping algebra.

Delius et al. [4, 5] have also looked for quantum deformations of Lie algebras in the ad-invariant subspaces of quantised enveloping algebras. Their quantum Lie algebras lack the antisymmetry and Jacobi axioms that are given here; on the other hand, they retain more of the symmetry of classical Lie algebras. Majid [8] has proposed axioms for “braided Lie algebras”; these include a Jacobi identity but no anticommutativity axiom and, not being satisfied in the classical case, do not constitute a generalisation of the Lie algebra axioms.

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Notation. I use the usual notation for coproducts in a Hopf algebra:

$$\begin{aligned}\Delta(x) &= \sum x_{(1)} \otimes x_{(2)}, \\ (\Delta \otimes \text{id}) \circ \Delta(x) &= \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \quad \text{etc.},\end{aligned}$$

and for the q -analogue function:

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

2 General Construction of Quantum Lie Algebras

Definition 2.1. A *quantum Lie algebra* over a field \mathbb{K} is a \mathbb{K} -vector space L together with linear maps $\beta : L \otimes L \rightarrow L$ (the quantum Lie bracket, written $x \otimes y \mapsto [x, y]$) and $\gamma : L \otimes L \rightarrow L \otimes L$ (the quantum antisymmetriser) satisfying:

1. Quantum antisymmetry: For $t \in L \otimes L$,

$$\gamma(t) = 0 \Rightarrow \beta(t) = 0. \quad (2.1)$$

2. The quantum Jacobi identity: For $x \in L$, define $\text{ad } x : L \rightarrow L$ by $\text{ad } x(y) = [x, y]$. Then

$$\text{ad}[x, y] = m \circ (\text{ad} \otimes \text{ad}) \circ \gamma(x \otimes y) \quad (2.2)$$

where $m : \text{End } L \otimes \text{End } L \rightarrow \text{End } L$ denotes multiplication of linear maps on L .

Definition 2.2. A quantum Lie algebra is *balanced* if it satisfies a second Jacobi identity,

3. The right quantum Jacobi identity: For $x \in L$, define $\text{rad } x : L \rightarrow L$ by $\text{rad } x(y) = [y, x]$. Then

$$\text{rad}[x, y] = \overleftarrow{m} \circ (\text{rad} \otimes \text{rad}) \circ \gamma(x \otimes y) \quad (2.3)$$

where $\overleftarrow{m} : \text{End } L \otimes \text{End } L \rightarrow \text{End } L$ denotes the opposite multiplication of linear maps: $\overleftarrow{m}(A \otimes B) = BA$.

If γ is the usual antisymmetrisation map, $\gamma(x \otimes y) = x \otimes y - y \otimes x$, the above become the usual antisymmetry and Jacobi axioms for a Lie algebra. These generalised axioms were found by Woronowicz to hold for the geometrical Lie bracket of left-invariant vector fields on a quantum Lie group [14]; he also found that in this case the quantum antisymmetriser was of the form $\gamma = 1 - \sigma$ where σ satisfies the braid relation. This is not true of the quantum Lie algebras described in this paper; I do not know if there is a weaker axiom of this type that they satisfy.

Woronowicz included only one Jacobi identity among his axioms; the geometrical construction does not obviously yield a balanced quantum Lie algebra. This is also, at first sight, true of the construction from universal enveloping algebras to be described in this paper. However, inspection of the quantum Lie algebra $\mathfrak{sl}(2)_q$ reveals that it is in fact balanced, and it seems likely that this is true more generally.

Definition 2.3. A *representation* of a quantum Lie algebra (L, β, γ) is a linear map $\rho : L \rightarrow \text{End } V$ for some vector space V , such that

$$\rho \circ \beta = m_V \circ (\rho \otimes \rho) \circ \gamma : L \otimes L \rightarrow \text{End } V \quad (2.4)$$

where m_V denotes multiplication of linear operators on V .

Then the Jacobi identity (2.3) states that ad is a representation of L in $\text{End}_{\mathbb{K}} V$, as with classical Lie algebras.

Definition 2.4. The *universal enveloping algebra* of a quantum Lie algebra (L, β, γ) is the quotient of the tensor algebra of L by the ideal generated by $(\beta - \gamma)(V \otimes V)$. Then every representation of L automatically generates a representation of its universal enveloping algebra.

Let x_1, \dots, x_n be a basis for L , and let γ_{ij}^{kl} be the matrix elements of γ with respect to the basis $x_i \otimes x_j$ of $L \otimes L$; in the classical case,

$$\gamma_{ij}^{kl} = \delta_i^k \delta_j^l - \delta_i^l \delta_j^k.$$

Then the Jacobi identity can be written as

$$[[x_i, x_j], x_k] = \gamma_{ij}^{lm} [x_l, [x_m, x_k]],$$

the right Jacobi identity as

$$[x_i, [x_j, x_k]] = \gamma_{jk}^{lm} [[x_i, x_l], x_m],$$

the representation property as

$$\rho([x_i, x_j]) = \gamma_{ij}^{kl} \rho(x_k) \rho(x_l)$$

and the relations in the universal enveloping algebra as

$$\gamma_{ij}^{kl} x_k x_l = [x_i, x_j] = \beta_{ij}^k x_k$$

where β_{ij}^k are the structure constants of L (i.e. the matrix elements of β).

Let q be a fixed element of the ground field \mathbb{K} , \mathfrak{g} a simple Lie algebra over \mathbb{K} , and let $\mathcal{U} = \overline{\mathcal{U}}_q(\mathfrak{g})$ be the simply-connected quantised enveloping algebra of \mathfrak{g} , which is defined as follows. \mathcal{U} contains a copy of the group algebra of the weight lattice of \mathfrak{g} (with basis elements denoted by q^λ where λ is an integral weight of \mathfrak{g} with respect to a Cartan subalgebra \mathfrak{h}) and generators

$E_1, \dots, E_r, F_1, \dots, F_r$ corresponding to fundamental roots H_1, \dots, H_r of \mathfrak{g} , with relations

$$\begin{aligned} q^\lambda E_i q^{-\lambda} &= q^{\langle H_i, \lambda \rangle} E_i \\ q^\lambda F_i q^{-\lambda} &= q^{-\langle H_i, \lambda \rangle} F_i \\ E_i F_j - F_j E_i &= \delta_{ij} \left(\frac{q^{2H_i} - q^{-2H_i}}{q_i - q_i^{-1}} \right) \end{aligned}$$

$$\begin{aligned} [E_i, [E_i, \dots, [E_i, E_j]_{q_i^{-n+1}}]_{q_i^{-n+3}} \dots]_{q_i^{n-1}} &= 0 \\ [F_i, [F_i, \dots, [F_i, F_j]_{q_i^{-n+1}}]_{q_i^{-n+3}} \dots]_{q_i^{n-1}} &= 0 \end{aligned}$$

where \langle, \rangle denotes the Killing form in the Cartan subalgebra \mathfrak{h} (which we do not distinguish from its dual \mathfrak{h}^*),

$$q_i = q^{\langle H_i, H_i \rangle},$$

$$n = \frac{2\langle H_i, H_j \rangle}{\langle H_i, H_i \rangle} = \text{number of roots } H_j + kH_i \text{ with } k > 0,$$

and $[X, Y]_p = pXY - p^{-1}YX.$

Then \mathcal{U} is a Hopf algebra with comultiplication

$$\begin{aligned} \Delta(E_i) &= E_i \otimes q^{-H_i} + q^{H_i} \otimes E_i, \\ \Delta(F_i) &= F_i \otimes q^{-H_i} + q^{H_i} \otimes F_i, \\ \Delta(q^\lambda) &= q^\lambda \otimes q^\lambda \end{aligned}$$

and antipodes

$$S(E_i) = -q_i^{-1} E_i, \quad S(F_i) = -q_i F_i, \quad S(q^\lambda) = q^{-\lambda}.$$

As in any Hopf algebra, we have the adjoint representation of \mathcal{U} on itself, given by $x \mapsto \text{ad } x \in \text{End}_{\mathbb{K}} \mathcal{U}$ where

$$\text{ad } x(y) = \sum x_{(1)} y S(x_{(2)}) \quad \text{if} \quad \Delta(x) = \sum x_{(1)} \otimes x_{(2)}.$$

This is a representation:

$$\text{ad}(xy) = \text{ad } x \cdot \text{ad } y, \tag{2.5}$$

and each $\text{ad } x$ is a generalised derivation of U , in the sense that

$$\text{ad } x(yz) = \sum \text{ad } x_{(1)}(y) \cdot \text{ad } x_{(2)}(z). \quad (2.6)$$

The adjoint action of the generators of U is given by

$$\begin{aligned} \text{ad } E_i(x) &= E_i x q^{H_i} - q^{H_i-1} x E_i, \\ \text{ad } F_i(x) &= F_i x q^{H_i} - q^{H_i+1} x F_i, \\ \text{ad } q^\lambda(x) &= q^\lambda x q^{-\lambda}. \end{aligned} \quad (2.7)$$

We use the adjoint representation to define a bracket on $\overline{U}_q(\mathfrak{g})$, defining

$$[x, y] = \text{ad } x(y). \quad (2.8)$$

Each $\text{ad } x$ is also a generalised derivation of this bracket:

$$[x, [y, z]] = \sum [[x_{(1)}, y], [x_{(2)}, z]]. \quad (2.9)$$

This is a kind of Jacobi identity for the adjoint bracket. The representation property gives another kind of Jacobi identity:

$$[[x, y], z] = \sum [x_{(1)}, [y, [x_{(2)}, z]]]. \quad (2.10)$$

Both of these are valid in any Hopf algebra [8]. However, they are not suitable as replacements for the Jacobi identity in defining a notion of quantum Lie algebra, as the ad-invariant subspaces of quantised enveloping algebras which are candidates for quantum Lie algebras are not usually subcoalgebras, and so the coproduct which is required to formulate the above Jacobi identities is not an intrinsic notion. That is why we are interested in the third kind of Jacobi identity, of the form (2.3), which is specific to quantised enveloping algebras.

The vector space $\overline{U}_q(\mathfrak{g})$ does not split up nicely into irreducible ad-invariant subspaces; strictly speaking it is not a direct sum of such spaces, for there are elements with components in an infinite number of them. The *locally finite part* of $\mathcal{U} = \overline{U}_q(\mathfrak{g})$ is the set of elements for which this is not true, i.e.

$$\mathcal{F} = \text{LF}\overline{U}_q(\mathfrak{g}) = \{x \in \mathcal{U} : \text{ad } \mathcal{U}(x) \text{ is finite-dimensional}\} \quad (2.11)$$

The discovery of Joseph and Letzter was that \mathcal{F} is easily described, and has the same composition as the classical enveloping algebra $U(\mathfrak{g})$:

Theorem 2.1 (Joseph and Letzter [6]). *Let λ be a dominant integral weight of \mathfrak{g} , V_λ the simple \mathfrak{g} -module with highest weight λ ; V_λ can also be regarded as a simple \mathcal{U} -module where $\mathcal{U} = \overline{U}_q(\mathfrak{g})$, and $\text{End}_{\mathbb{K}} V_\lambda$ is a \mathcal{U} -module by conjugation (i.e. if $x \in \mathcal{U}$ acts on V_λ by $\rho(x)$, it acts on $\text{End}_{\mathbb{K}} V_\lambda$ by $T \mapsto \sum \rho(x_{(1)})T\rho(Sx_{(2)})$). Then*

1. $E_\lambda = \text{ad}\mathcal{U}(q^{-4\lambda})$ is a finite-dimensional ad-invariant subspace of \mathcal{U} , isomorphic to $\text{End}_{\mathbb{K}} V_\lambda$ as \mathcal{U} -module.
2. The locally finite part of \mathcal{U} is

$$\mathcal{F} = \sum_{\lambda} E_{\lambda} \quad (2.12)$$

where the sum extends over all dominant integral weights of \mathfrak{g} .

For a dominant integral weight λ , we write $N_\lambda = \dim_{\mathbb{K}} V_\lambda$. Then our result on the existence of quantum Lie algebras in $\overline{U}_q(\mathfrak{g})$ is

Theorem 2.2. *For each dominant integral weight λ , $\overline{U}_q(\mathfrak{g})$ contains an $(N_\lambda^2 - 1)$ -dimensional ad-invariant subspace L_λ , a linear map $\sigma : L_\lambda \otimes L_\lambda \rightarrow L_\lambda \otimes L_\lambda$ and a central element $C_\lambda \in \overline{U}_q(\mathfrak{g})$ such that*

$$xy - m \circ \sigma(x \otimes y) = C_\lambda[x, y] \quad (x, y \in L_\lambda) \quad (2.13)$$

$$\text{i.e.} \quad x_i x_j - \sigma_{ij}^{kl} x_k x_l = C_\lambda[x_i, x_j] \quad (2.14)$$

where m denotes multiplication in $\overline{U}_q(\mathfrak{g})$, $[\cdot, \cdot]$ is the adjoint bracket, and $\{x_i\}$ is a basis of L_λ .

Proof. Writing $\mathcal{U} = \overline{U}_q(\mathfrak{g})$, let $\overline{L} = \text{ad}\mathcal{U}(q^{-4\lambda})$. Then \overline{L} is invariant under $\text{ad}\mathcal{U}$, and by Theorem 2.1 it is isomorphic as a \mathcal{U} -module to $\text{End}_{\mathbb{K}} V_\lambda$. This contains a one-dimensional submodule (spanned by the identity of $\text{End}_{\mathbb{K}} V_\lambda$) and so by complete reducibility of representations of $\overline{U}_q(\mathfrak{g})$ [3, p. 324] we can write

$$\overline{L} = \mathbb{K}C_\lambda \oplus L$$

as a \mathcal{U} -module, for some $C_\lambda \in \overline{L}$. Then C_λ carries the trivial representation of \mathcal{U} :

$$\text{ad } x(C_\lambda) = \varepsilon(x)C_\lambda, \quad \forall x \in \mathcal{U}.$$

Hence C_λ is a central element of \mathcal{U} , for

$$xC_\lambda = \sum x_{(1)}C_\lambda Sx_{(2)}x_{(3)} = \sum \text{ad } x_{(1)}(C_\lambda)x_{(2)} = \sum \varepsilon(x_{(1)})C_\lambda x_{(2)} = C_\lambda x.$$

Now \overline{L} is a left coideal of \mathcal{U} :

$$\begin{aligned}
x \in \overline{L} &\implies x = \text{ad } u(q^{-4\lambda}) \text{ for some } u \in \mathcal{U} \\
&\implies \Delta(x) = \sum \Delta(u_{(1)})\Delta(q^{-4\lambda})\Delta(Su_{(2)}) \\
&= \sum (u_{(1)} \otimes u_{(2)})(q^{-4\lambda} \otimes q^{-4\lambda})(Su_{(4)} \otimes Su_{(3)}) \\
&\in \mathcal{U} \otimes \overline{L}.
\end{aligned} \tag{2.15}$$

Hence for any $x \in L$ we can write

$$\Delta(x) = x_0 \otimes C_\lambda + \sum u' \otimes x'$$

with $x_0, u' \in U$ and $x' \in L$. In fact we can choose C_λ so that $x_0 = x$: for if $q^{-4\lambda} = C_\lambda + w$ with $w \in L$ and if $x = \text{ad } u(q^{-4\lambda})$, then from (2.15)

$$\begin{aligned}
\Delta(x) &= \sum u_{(1)}q^{-4\lambda}Su_{(3)} \otimes \text{ad } u_{(2)}(C_\lambda + w) \\
&= \sum u_{(1)}q^{-4\lambda}Su_{(3)} \otimes (\varepsilon(u_{(2)})C_\lambda + \text{ad } u_{(2)}(w)) \\
&= u_{(1)}q^{-4\lambda}Su_{(2)} \otimes C_\lambda + \sum u_{(1)}q^{-4\lambda}Su_{(3)} \otimes \text{ad } u_{(2)}(w)
\end{aligned}$$

but $\text{ad } u_{(2)}(w) \in L$ since L is invariant under $\text{ad } U$, and the first term is $x \otimes C_\lambda$.

In L we have the bracket $[x, y] = \text{ad } x(y)$. Define $\sigma : L \otimes L \rightarrow L \otimes L$ by

$$\sigma(x \otimes y) = \sum \text{ad } u'(y) \otimes x' \quad \text{where } \Delta(x) = x \otimes C_\lambda + \sum u' \otimes x' \tag{2.16}$$

$$\text{i.e.} \quad \sigma(x \otimes y) = \sum \text{ad } x_{(1)}(y) \otimes x_{(2)} - [x, y] \otimes C_\lambda. \tag{2.17}$$

Then

$$\begin{aligned}
m \circ \sigma(x \otimes y) &= \sum x_{(1)}y(Sx_{(2)})x_{(3)} - [x, y]C_\lambda \\
&= xy - [x, y]C_\lambda.
\end{aligned}$$

Since C_λ is central, this establishes the theorem. \square

The existence of a quantum Lie algebra in $U_q(\mathfrak{sl}(n))$ follows immediately:

Theorem 2.3. *If q is not a 2nth root of unity nor a primitive twelfth root of unity, $U_q(\mathfrak{sl}(n))$ contains an $(n^2 - 1)$ -dimensional quantum Lie algebra with the adjoint bracket.*

Proof. In Theorem 2, take $\mathfrak{g} = \mathfrak{sl}(n)$ and λ the highest weight of the n -dimensional (defining) representation. Then L_λ carries an irreducible representation of $U_q(\mathfrak{g})$ (a deformation of the adjoint representation of $\mathfrak{sl}(n)$). Since C_λ is central, $\text{ad } C_\lambda$ acts on L_λ as a multiple of the identity; in a lemma we will show that this multiple is $q^2 - 1 + q^{-2}$ (which vanishes only if q is a primitive 12th root of unity). Assuming this for the moment, we apply ad to (2.14) and restrict to $L = L_\lambda$ to obtain

$$\text{ad}_L[x_i, x_j] = \gamma_{ij}^{kl} \text{ad}_L x_k \cdot \text{ad}_L x_l \quad (2.18)$$

where

$$\gamma = \frac{1 - \sigma}{q^2 - 1 + q^{-2}}.$$

Thus the quantum Jacobi identity is satisfied in L by the adjoint bracket. Also, from 2.13,

$$\begin{aligned} \gamma(x \otimes y) = 0 &\implies \sigma\left(\sum x \otimes y\right) = \sum x \otimes y \\ &\implies C_\lambda \sum [x, y] = 0 \\ &\implies \sum [x, y] = 0 \end{aligned}$$

since $\overline{U}_q(\mathfrak{g})$ contains no zero divisors [6]. Thus the adjoint bracket restricted to L_λ has the antisymmetry property with respect to the quantum antisymmetriser γ .

Completion of the proof now needs the following lemmas.

Lemma 2.1. *The central element in \overline{L} is given by*

$$C_\lambda = \sum_{r=0}^{n-1} (-1)^r \frac{[n-r]_q}{[n]_q} K_r \quad (2.19)$$

where K_r is defined recursively by

$$K_r = \text{ad}(F_r E_r) K_{r-1}, \quad K_0 = q^{-4\lambda}. \quad (2.20)$$

Lemma 2.2. *The adjoint action of the central element on L is*

$$\text{ad } C_\lambda(x) = (q^2 - 1 + q^{-2})x \quad \text{for } x \in L. \quad (2.21)$$

The proofs, both unenlightening calculations, are relegated to an appendix. \square

Although the antisymmetriser γ in this Lie algebra is not of the Woronowicz form $\gamma = 1 - \sigma$ where σ satisfies the braid relation, it is of a related form. According to 2.18 and 2.17, γ is a scalar multiple of $1 - \sigma$ where

$$\sigma(x \otimes y) = \bar{\sigma}(x \otimes y) - [x, y] \otimes C_\lambda$$

and $\bar{\sigma} : \bar{L} \otimes \bar{L} \rightarrow \bar{L} \otimes \bar{L}$ is given by

$$\bar{\sigma}(x \otimes y) = \sum \text{ad } x_{(1)}(y) \otimes x_{(2)}.$$

It is a straightforward exercise in Hopf algebra manipulation [15] to show that $\bar{\sigma}$ satisfies the braid relation.

3 The quantum Lie algebra $\mathfrak{sl}(2)_q$

In the case of $\mathfrak{g} = \mathfrak{sl}(2)$, the simply connected quantised enveloping algebra is generated by E , F and $q^{\pm \frac{1}{2}H}$ with relations

$$\begin{aligned} q^H E q^{-H} &= qE, \\ q^H F q^{-H} &= q^{-1}F, \\ EF - FE &= \frac{q^{2H} - q^{-2H}}{q - q^{-1}}. \end{aligned}$$

The highest weight of the fundamental two-dimensional representation is (i.e. corresponds via the Killing form to) $\lambda = \frac{1}{2}H$, so in the notation of Lemma 2.1 we have

$$\begin{aligned} K_0 &= q^{-4\lambda} = q^{-2H}, \\ K_1 &= \text{ad}(FE)K_0 = -(q - q^{-1})(qEF - q^{-1}FE). \end{aligned}$$

Hence, according to Lemma 2.1, the central element is

$$C = q^{-2H} + \left(\frac{q - q^{-1}}{q + q^{-1}} \right) (qEF - q^{-1}FE) \quad (3.1)$$

A basis for the quantum Lie algebra is (X_\pm, X_0) where

$$\begin{aligned} X_+ &= \frac{\text{ad } E(q^{-2H})}{q - q^{-1}} = q^{-H}E, \\ X_- &= -\frac{\text{ad } F(q^{-2H})}{q - q^{-1}} = q^{-H}F, \\ X_0 &= -\frac{\text{ad}(FE)q^{-2H}}{q^2 - q^{-2}} = \frac{qEF - q^{-1}FE}{q + q^{-1}} \\ &= \frac{C - q^{-2H}}{q - q^{-1}}. \end{aligned} \quad (3.2)$$

The adjoint brackets of these basis elements can be calculated to be

$$\begin{aligned}
[X_+, X_+] &= 0, & [X_+, X_0] &= -q^{-1}X_+, & [X_+, X_-] &= (q + q^{-1})X_0, \\
[X_0, X_+] &= qX_+, & [X_0, X_0] &= (q - q^{-1})X_0, & [X_0, X_-] &= -q^{-1}X_-, \\
[X_-, X_+] &= -(q + q^{-1})X_0, & [X_-, X_0] &= qX_-, & [X_-, X_-] &= 0.
\end{aligned} \tag{3.3}$$

The quantum Lie algebra $\mathfrak{sl}(2)_q$ is defined by these brackets together with the quantum antisymmetriser $\gamma = (q^2 - 1 + q^{-2})^{-1}\gamma'$ where

$$\begin{aligned}
\gamma'(X_\pm \otimes X_\pm) &= 0, \\
\gamma'(X_\pm \otimes X_0) &= q^{\mp 2}X_\pm \otimes X_0 - X_0 \otimes X_\pm, \\
\gamma'(X_0 \otimes X_\pm) &= q^{\pm 2}X_0 \otimes X_\pm - X_\pm \otimes X_0, \\
\gamma'(X_+ \otimes X_-) &= X_+ \otimes X_- - X_- \otimes X_+ + (q^2 - q^{-2})X_0 \otimes X_0, \\
\gamma'(X_- \otimes X_+) &= -\gamma'(X_+ \otimes X_-), \\
\gamma'(X_+ \otimes X_0) &= (q - q^{-1})^2 X_0 \otimes X_0 + \left(\frac{q - q^{-1}}{q + q^{-1}} \right) (X_+ \otimes X_- - X_- \otimes X_+) \\
&= \left(\frac{q - q^{-1}}{q + q^{-1}} \right) \gamma'(X_+ \otimes X_-).
\end{aligned} \tag{3.4}$$

We note the following properties of $\mathfrak{sl}(2)_q$:

Proposition 1. 1. $\mathfrak{sl}(2)_q$ is a balanced quantum Lie algebra.

2. The quantum antisymmetriser γ is essentially idempotent:

$$\gamma^2 = \left(\frac{q^2 + q^{-2}}{q^2 - 1 + q^{-2}} \right) \gamma.$$

Proof. By calculation. □

The second of these properties is peculiar to $\mathfrak{sl}(2)_q$, reflecting the simple structure of its representations, as will become apparent when we consider $\mathfrak{sl}(3)_q$. The first, however, may be more general.

The enveloping algebra of the quantum Lie algebra $U(\mathfrak{sl}(2)_q)$ — not to be confused with the quantised enveloping algebra $U_q(\mathfrak{sl}(2))$ — is defined by means of the brackets (3.3) and the antisymmetriser (3.4). By redefining the generators to absorb the factor $q^2 - 1 + q^{-2}$, it can be presented as the

associative algebra generated by three elements Y_0, Y_\pm with relations

$$\begin{aligned} qY_0Y_+ - q^{-1}Y_+Y_0 &= Y_+ \\ q^{-1}Y_0Y_- - qY_-Y_0 &= -Y_- \\ Y_+Y_- - Y_-Y_+ + (q^2 - q^{-2})Y_0^2 &= (q + q^{-1})Y_0. \end{aligned} \quad (3.5)$$

On the other hand, as elements of the quantised enveloping algebra $U_q(\mathfrak{sl}(2))$ the generators X_\pm, X_0 and the central element C satisfy (by Theorem 2.2)

$$\begin{aligned} q^2X_0X_+ - X_+X_0 &= qCX_+, \\ q^{-2}X_0X_- - X_-X_0 &= -q^{-1}CX_-, \\ X_+X_- - X_-X_+ + (q^2 - q^{-2})X_0^2 &= (q + q^{-1})CX_0 \\ \text{and} \quad CX_m &= X_mC \quad (m = 0, \pm 1). \end{aligned} \quad (3.6)$$

However, the central element C is not independent of the other generators X_m : there is a further relation

$$C^2 = (q - q^{-1})^2 \left(X_0^2 + \frac{qX_-X_+ + q^{-1}X_+X_-}{q + q^{-1}} \right) + 1 \quad (3.7)$$

as can be verified by direct calculation.

The main results of this section are that the locally finite part of $\overline{U}_q(\mathfrak{g})$ is isomorphic, as an algebra, to the algebra with generators X_0, X_\pm, C and relations (3.6), and as a vector space to the algebra of polynomials in four commuting variables X_0, X_\pm, C subject to the relation (3.7). We use the following notation for the various algebras:

- \mathcal{A} = enveloping algebra of the quantum Lie algebra $\mathfrak{sl}(2)_q$
= algebra generated by Y_\pm, Y_0 with relations (3.5),
- \mathcal{B} = algebra generated by X_\pm, X_0, C with relations (3.6),
- \mathcal{C} = algebra generated by X_\pm, X_0, C with relations (3.6) and (3.7),
- \mathcal{F} = locally finite part of $\overline{U}_q(\mathfrak{g})$.

Then \mathcal{A}, \mathcal{B} and \mathcal{C} are related by homomorphisms

$$\begin{aligned} \varphi &: \mathcal{B} \rightarrow \mathcal{A} \quad \text{with } \varphi(C) = 1 \\ \psi &: \mathcal{B} \rightarrow \mathcal{C} \quad \text{enforcing (3.7).} \end{aligned}$$

Lemma 3.1. *The algebra \mathcal{A} has a basis of ordered monomials $Y_-^l Y_0^m Y_+^n$ where l, m, n are non-negative integers.*

Proof. It is evident that the commutation relations (3.5) allow any product of Y s to be expressed as a combination of ordered monomials. We have only to show that these monomials are independent. According to the diamond lemma [1], this will follow if it can be shown that the two methods of reducing $Y_+Y_0Y_-$ to a combination of ordered monomials both lead to the same result. This is easily checked using (3.5). \square

Lemma 3.2. *The algebra \mathcal{B} has a basis of ordered monomials $C^k X_-^l X_0^m X_+^n$.*

Proof. $\mathcal{A} \otimes \mathbb{K}[C]$ contains an isomorphic copy of \mathcal{B} generated by C and $X_m = CY_m$. This is spanned by monomials $C^k X_-^l X_0^m X_+^n = C^{k+l+m+n} Y_-^l Y_0^m Y_+^n$ and by Lemma 3.1 these are independent. \square

Lemma 3.3. *The algebra \mathcal{C} is isomorphic as a vector space to the space of polynomials in four commuting variables C, X_0, X_+, X_- subject to the relation (3.7).*

Proof. Let

$$C_2 = (q + q^{-1})X_0^2 + qX_-X_+ + q^{-1}X_+X_-.$$

Using (3.6), it can be checked that C_2 commutes with X_0, X_+ and X_- . Hence the two-sided ideal I generated by $(q + q^{-1})(C^2 - 1) - (q - q^{-1})^2 C_2$ (i.e. the kernel of the relation (3.7)) is the same as the left ideal generated by it. It follows that the monomials $X_-^l X_0^m X_+^n$ and $CX_-^l X_0^m X_+^n$ are independent modulo I , and therefore form a basis of \mathcal{C} . But these monomials also form a basis of the commutative algebra described in the theorem. \square

Theorem 3.1. *The locally finite part of $\overline{U}_q(\mathfrak{sl}(2))$ is isomorphic to \mathcal{C} .*

Proof. By Theorem 2.1, the locally finite part of $\mathcal{U} = \overline{U}_q(\mathfrak{g})$ is

$$\mathcal{F} = \sum_{n=0}^{\infty} V_n \quad \text{where} \quad V_n = \text{ad } \mathcal{U}(q^{-2nH}).$$

Since $\frac{1}{2}nH$ is the highest weight of the $(n+1)$ -dimensional representation of \mathcal{U} , $\dim V_n = (n+1)^2$. By the derivation property (2.6) of the adjoint action,

$$V_{m+n} \subseteq V_m V_n.$$

Hence

$$V_n \subseteq V_1^n.$$

Let \mathcal{U}' be the subalgebra of \mathcal{U} generated by V_1 , i.e. by C , X_- , X_+ and X_0 . Since the relations (3.6) and (3.7) hold in \mathcal{U} , \mathcal{U}' is a quotient of \mathcal{C} and therefore, by Lemma 3.3, is spanned by monomials $X_-^k X_0^l X_+^m$ and $C X_-^k X_0^l X_+^m$. Let W_n be the subspace spanned by monomials $X_-^k X_0^l X_+^m$ with $k + l + m = n$. Then

$$V_1^{n+2} = V_1^n + C W_{n+1} + W_{n+2} \quad (3.8)$$

as can be proved by induction, using

$$V_1 W_r = C W_r + W_{r+1}$$

and

$$C^2 W_r + W_{r+2} = W_r + W_{r+2}$$

which follows from (3.7). But

$$\dim W_r \leq \frac{1}{2}(r+1)(r+2), \quad (3.9)$$

so

$$\dim(V_1^{n+2}) \leq \dim(V_1^n) + (n+3)^2.$$

We can now prove by induction that

$$V_1^n = V_n \oplus V_{n-2} \oplus \cdots \oplus (V_0 \text{ or } V_1).$$

For if this is true, then from (3.8) we have

$$V_1^{n+2} \supseteq V_{n+2} \oplus V_n \oplus \cdots$$

since the V_n are known to be complementary in \mathcal{U} [6]. By dimensions the inclusion must be equality, so the sum in (3.8) is direct and also (3.9) must be equality. It follows that the monomials $X_-^k X_0^l X_+^m$ and $C X_-^k X_0^l X_+^m$ are independent and form a basis of $\sum V_n = \mathcal{F}$. By Lemma 3.3, \mathcal{F} is isomorphic to \mathcal{C} . \square

In one sense, the locally finite part of the quantised enveloping algebra is the same size as the classical enveloping algebra of $\mathfrak{sl}(2)$. Both can be written as $Z \otimes \mathcal{U}_0$ where Z , the centre, is a polynomial algebra in one variable and \mathcal{U}_0 has the same decomposition into simple modules under the adjoint action in both cases. However, the above shows that it is more appropriate to regard the locally finite part of $\overline{\mathcal{U}}_q(\mathfrak{sl}(2))$ as twice as big as the classical enveloping algebra. In the classical case the centre is generated by a Casimir element which is a polynomial in the Lie algebra generators; in the quantum case the centre is generated by a square root of such a polynomial, tending to 1 in the classical limit.

4 The Quantum Lie Algebra $\mathfrak{sl}(3)_q$

This section contains some details of and observations on the quantum Lie algebra of type A_2 . All were established by mindless calculation.

We take λ to be the highest weight of the triplet representation,

$$\lambda = \frac{1}{3}(2H_1 + H_2)$$

(identifying weights with elements of the Cartan subalgebra by means of the Killing form, and taking H_1 and H_2 to have unit length). A convenient basis for the quantum Lie algebra $\mathfrak{sl}(3)_q$ is the following, consisting of two elements T_1, T_2 (scalar multiples of the K_1, K_2 defined in Lemma 2.1) and elements $X_\alpha = X_{\pm 1}, X_{\pm 2}, X_{\pm 12}$ corresponding to the roots $\alpha = \pm H_1, \pm H_2, \pm(H_1 + H_2)$ of $\mathfrak{sl}(3)$:

$$\begin{aligned} T_1 &= \frac{\text{ad}(F_1 E_1) q^{-4\lambda}}{q - q^{-1}} = q^{-\frac{2}{3}(H_1 + 2H_2)}(q E_1 F_1 - q^{-1} F_1 E_1) \\ T_2 &= \text{ad}(F_2 E_2) T_1 \\ X_1 &= \frac{\text{ad } E_1(q^{-4\lambda})}{q - q^{-1}} = q^{-\frac{1}{3}(5H_1 + 4H_2)} E_1 \\ X_{12} &= \text{ad } E_2(X_1) = q^{-\frac{1}{3}(5H_1 + H_2)}(E_2 E_1 - q^{-1} E_1 E_2) \\ X_2 &= \text{ad } F_1(X_{12}) \\ X_{-1} &= \frac{\text{ad } F_1(q^{-4\lambda})}{q - q^{-1}} = -q^{-\frac{1}{3}(5H_1 + 4H_2)} F_1 \\ X_{-12} &= \text{ad } F_2(X_{-1}) = q^{-\frac{1}{3}(5H_1 + H_2)}(q F_1 F_2 - F_2 F_1) \\ X_{-2} &= -\text{ad } E_1(X_{-12}) \end{aligned}$$

The adjoint action of $\overline{U}_q(\mathfrak{g})$ on these basis elements is:

$$\begin{aligned} \text{ad}(q^{H_i}) T_j &= T_j \\ \text{ad}(q^{H_i}) X_{\pm j} &= \begin{cases} q^{\pm 1} X_{\pm j} & \text{if } i = j \\ q^{\mp 1/2} X_{\pm j} & \text{if } i \neq j \end{cases} \\ \text{ad}(q^{H_i}) X_{\pm 12} &= q^{\pm 1/2} X_{\pm 12} \\ \text{ad } E_i(X_\alpha) &= \begin{cases} X_{\alpha + H_i} & \text{if } \alpha + H_i \text{ is a root} \\ T_i & \text{if } \alpha = -H_i \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$\text{ad } F_i(X_\alpha) = \begin{cases} X_{\alpha-H_i} & \text{if } \alpha - H_i \text{ is a root} \\ T_i & \text{if } \alpha = H_i \\ 0 & \text{otherwise} \end{cases}$$

$$\text{ad } E_i(T_j) = \begin{cases} (q + q^{-1})X_i & \text{if } i = j \\ X_{12} & \text{if } i \neq j \end{cases}$$

$$\text{ad } F_i(T_j) = \begin{cases} (q + q^{-1})X_{-i} & \text{if } i = j \\ X_{-12} & \text{if } i \neq j \end{cases}$$

From this the adjoint brackets between the basis elements can be calculated; the result is given in the table, in which $[X, Y]$ is given in the row labelled by X and the column labelled by Y . If $\alpha + \beta \neq 0$, $[X_\alpha, X_\beta] = c_{\alpha\beta}X_{\alpha+\beta}$ is represented by the coefficient $c_{\alpha\beta}$; similarly, $[T_i, X_\alpha] = b_{i\alpha}X_\alpha$ is represented by $b_{i\alpha}$.

	T_1	T_2	X_1	X_{-1}
T_1	$-(q^2 - q^{-2})T_1$	$-(q - q^{-1})T_1$	$-q(q + q^{-1})$	$q^{-1}(q + q^{-1})$
T_2	$-(q - q^{-1})T_1$		$-q$	q^{-1}
X_1	$q^{-1}(q + q^{-1})$	q^{-1}	0	T_1
X_{-1}	$-q(q + q^{-1})$	$-q$	$-T_1$	0
X_2	$-q^2$	$-q^{-1}(q^2 + q^{-2})$	$-q^{3/2}$	0
X_{-2}	q^{-2}	$q(q^2 + q^{-2})$	0	$q^{-3/2}$
X_{12}	1	$-q^{-3}$	0	$q^{1/2}$
X_{-12}	-1	q^3	$-q^{-1/2}$	0

	X_2	X_{-2}	X_{12}	X_{-12}
T_1	q^{-2}	$-q^2$	-1	1
T_2	$q(q^2 + q^{-2})$	$-q^{-1}(q^2 + q^{-2})$	q^3	$-q^{-3}$
X_1	$q^{-3/2}$	0	0	$q^{1/2}$
X_{-1}	0	$-q^{3/2}$	$-q^{-1/2}$	0
X_2	0		0	$-q^{-5/2}$
X_{-2}		0	$q^{5/2}$	0
X_{12}	0	$-q^{-5/2}$	0	$-q^{-1}T_1 + T_2$
X_{-12}	$q^{5/2}$	0	$qT_1 - T_2$	0

Figure 1: The brackets of $\mathfrak{sl}(3)_q$.

The entries which are too long to fit into the table are

$$\begin{aligned}[T_2, T_2] &= -(q^2 - q^{-2})T_1 + (q^3 - q^{-3})T_2, \\ [X_2, X_{-2}] &= q^{-1}(q - q^{-1})T_1 - qT_2, \\ [X_{-2}, X_2] &= q(q - q^{-1})T_1 + q^{-1}T_2.\end{aligned}$$

Notice that if L is regarded as a module over $\mathbb{K}[q, q^{-1}]$ then, with this basis, the brackets exhibit the simple antisymmetry which was also found by Delius and Hüffmann. Define a map of q -conjugation on L , $X \rightarrow X^\nabla$, by the requirements that it leaves the above basis elements fixed and that $\{f(q)X\}^\nabla = f(q^{-1})X^\nabla$. Then it is apparent from the table that

$$[Y, X] = -[X, Y]^\nabla \quad \text{for all } X, Y \in L. \quad (4.1)$$

The quantum antisymmetriser of $\overline{U}_q(\mathfrak{g})$ can be described as follows. The quantum Lie algebra $L = \mathfrak{sl}(3)_q$ carries an irreducible representation of $\overline{U}_q(\mathfrak{g})$; as in the classical case, $L \otimes L$ splits into irreducible representations as

$$L \otimes L = M_{27} \oplus M_{10} \oplus M_{10}^* \oplus M_8^a \oplus M_8^s \oplus M_1$$

where the numerical labels 27, 10, 8, 1 indicate the dimensions of the simple modules. These are completely defined by their highest-weight elements:

$$\begin{aligned}W_{27} &= X_{12} \otimes X_{12}, \\ W_{10} &= q^{1/2}X_1 \otimes X_{12} - q^{1/2}X_{12} \otimes X_1, \\ W_{10}^* &= q^{1/2}X_2 \otimes X_{12} - q^{-1/2}X_{12} \otimes X_2, \\ W_8^s &= \{q^{5/2}(q + q^{-1}) + q^{-5/2}\} X_1 \otimes X_2 + \{q^{-5/2}(q + q^{-1}) + q^{5/2}\} X_2 \otimes X_1 \\ &\quad - (q^4T_1 + q^{-1}T_2) \otimes X_2 - X_2 \otimes (q^{-4}T_1 + qT_2), \\ W_8^a &= q^{3/2}X_1 \otimes X_2 - q^{-3/2}X_2 \otimes X_1 - q(qT_1 - T_2) \otimes X_{12} \\ &\quad + q^{-1}X_{12} \otimes (q^{-1}T_1 - T_2), \\ W_1 &= T_1 \otimes T_2 + T_2 \otimes T_1 - (q + q^{-1})(T_1 \otimes T_1 + T_2 \otimes T_2) \\ &\quad + (q^2 + 1 + q^{-2})(q^{-1}X_1 \otimes X_{-1} + qX_{-1} \otimes X_1 \\ &\quad + q^{-1}X_2 \otimes X_{-2} + qX_{-2} \otimes X_2 - q^{-2}X_{12} \otimes X_{-12} - q^2X_{-12} \otimes X_{12}).\end{aligned} \quad (4.2)$$

Let $\Pi_{27}, \Pi_{10}, \Pi_{10}^*, \Pi_8^s, \Pi_8^a$ and Π_1 denote the projectors onto these simple modules. Then

Lemma 4.1. *The quantum antisymmetriser of $\overline{U}_q(\mathfrak{g})$ is*

$$\gamma = \frac{(q^2 + 1)\Pi_{10}^* + (q^2 + q^{-2})\Pi_8^a + (1 + q^{-2})\Pi_{10}}{q^2 - 1 + q^{-2}} \quad (4.3)$$

Proof. Straightforward Hopf-algebra manipulations show that γ commutes with the diagonal adjoint action of $\mathcal{U} = U_q(\mathfrak{sl}(3))$ on $L \otimes L$, i.e. with $(\text{ad} \otimes \text{ad}) \circ \Delta(u)$ for all $u \in \mathcal{U}$. Thus it is only necessary to calculate the action of γ on the highest-weight elements listed above. This is not too laborious for the cases of W_{10}, W_{10}^* and W_{27} . For the octets the relevant highest-weight elements (eigenvectors of γ) and the eigenvalues can be determined from the Jacobi identity: in the two-dimensional space of highest-weight octet vectors (elements of $L \otimes L$ with weight $H_1 + H_2$ which are annihilated by $(\text{ad} \otimes \text{ad}) \circ \Delta(E_1)$ and $(\text{ad} \otimes \text{ad}) \circ \Delta(E_2)$), the eigenvectors of γ are the relative eigenvectors of $\text{ad} \circ \beta$ and $m \circ (\text{ad} \otimes \text{ad})$.

The eigenvalue 0 for the singlet can be verified in a similar way. We find that $\beta(W_1) = 0$ but $m \circ (\text{ad} \otimes \text{ad})(W_1) \neq 0$. Since $\text{ad} \circ \beta(W_1) = m \circ (\text{ad} \otimes \text{ad}) \circ \gamma(W_1)$ and $\gamma(W_1)$ must be a multiple of W_1 , it follows that $\gamma(W_1) = 0$. \square

The quantised enveloping algebra $\mathcal{U} = \overline{U}_q(\mathfrak{sl}(3))$ has a symmetry between (E_1, F_1, H_1) and (E_2, F_2, H_2) corresponding to the outer automorphism of $\mathfrak{sl}(3)$ which comes from the symmetry of the Dynkin diagram $\circ - \circ$. The quantum Lie algebra $\mathcal{L} = \mathfrak{sl}(3)_q = \text{ad} \mathcal{U}(q^{-4\lambda})$ is not invariant under this automorphism of \mathcal{U} . Indeed, the weight diagram of the 3-dimensional representation is taken by this symmetry to that of the conjugate representation, with highest weight $\lambda^* \neq \lambda$. This gives another quantum Lie algebra $L^* \subset \text{ad} \mathcal{U}(q^{-4\lambda^*})$ with $L^* \neq L$. This is another octet (under the adjoint representation) which also generates the locally finite part of \mathcal{U} . It is interesting to express the elements of L^* in terms of those of L and to compare this situation with the classical one.

In the classical enveloping algebra the octets are the Lie algebra itself, one which is quadratic in the generators, and multiples of these by functions of the Casimirs. The outer automorphism acts linearly on the generators. In the locally finite part of the quantised enveloping algebra we have the same general structure [6]; the quadratic octet has highest-weight element obtained by multiplying the factors of the tensor W_8^s in (4.2). The highest-weight element of the octet L is X_{12} . Let X_{12}^* be the highest-weight element of L^* , and let $Y_{12} = m(W_8^s)$. Then a calculation gives

$$X_{12}^* = \frac{q^{1/2}(q + q^{-1})CX_{12} + q(q - q^{-1})Y_{12}}{(q^{1/2} + q^{-1/2})(q^2 + 1 + q^{-2})}. \quad (4.4)$$

Thus the elements of L^* are quadratic in the generators L , and the quantum counterpart of the outer automorphism is a nonlinear symmetry of $\overline{U}_q(\mathfrak{sl}(3))$.

5 Appendix. Proofs of Lemmas 2.1 and 2.2

Lemma 2.1 The central element in \bar{L} is given by

$$C_\lambda = \sum_{r=0}^{n-1} (-1)^r \frac{[n-r]_q}{[n]_q} K_r \quad (2.19)$$

where K_r is defined recursively by

$$K_r = \text{ad}(F_r E_r) K_{r-1}, \quad K_0 = q^{-4\lambda}. \quad (2.20)$$

Proof. Let C be defined by (2.19). Since this has zero weight for the Cartan algebra generated by H_1, \dots, H_{n-1} (i.e. $\text{ad } q^{H_i}(C) = C$), in order to show that it is central it is sufficient to show that $\text{ad } E_i(C) = 0$ for $i = 1, \dots, n-1$. We calculate $\text{ad } E_i(K_j)$.

Since $\text{ad } \bar{U}_q(\mathfrak{g})(q^{-4\lambda})$ is the direct sum of the q -analogues of the scalar and the adjoint modules of $\mathfrak{sl}(n)$, the structure of the roots of $\mathfrak{sl}(n)$ gives

$$\text{ad } E_j \text{ad } E_i \text{ad } E_{i-1} \cdots \text{ad } E_1(q^{-4\lambda}) = 0 \quad \text{unless } j = i + 1. \quad (5.1)$$

Hence $\text{ad } E_j(K_i) = 0$ if $j > i + 1$ and $i > 0$.

This also holds if $i = 0$ since $q^{-4\lambda}$ commutes with E_j for $j > 1$.

For $i = j$ we have

$$\begin{aligned} \text{ad } E_i(K_i) &= \text{ad}(F_i E_i + [2H_i]_q) \text{ad } E_i(K_{i-1}) \\ &= [\langle 2H_i, H_i \rangle]_q \text{ad } E_i(K_{i-1}) \\ &= [2]_q \text{ad } E_i(K_{i-1}) \end{aligned}$$

since

$$\begin{aligned} (\text{ad } E_i)^2 K_{i-1} &= \text{ad } F_{i-1} \cdots \text{ad } F_1 (\text{ad } E_i)^2 \text{ad } E_{i-1} \cdots \text{ad } E_1(q^{-4\lambda}) \\ &= 0 \quad \text{by (5.1).} \end{aligned}$$

For $j = i - 1$ we have

$$\begin{aligned} \text{ad } E_j(K_{j+1}) &= \text{ad } F_{j+1} \text{ad}(F_j E_j + [2H_j]) \text{ad } E_{j+1} \text{ad } E_j(K_{j-1}) \\ &= \text{ad}([2H_j - 1]_q) \text{ad } F_{j+1} \text{ad } E_{j+1} \text{ad } E_j(K_{j-1}) \\ &= \text{ad}(F_{j+1} E_{j+1} E_j) K_{j-1} \quad \text{since } [2\langle H_j, H_j \rangle - 1]_q = 1 \\ &= \text{ad}[(E_{j+1} F_{j+1} - [2H_{j+1}]_q) E_j] K_{j-1} \\ &= \text{ad } E_j(K_{j-1}) \end{aligned} \quad (5.2)$$

since $\text{ad } F_{j+1} \text{ad } E_j(K_{j-1}) = 0$ by the structure of the roots of $\mathfrak{sl}(n)$.

For $j < i - 1$,

$$\begin{aligned}
\text{ad } E_j(K_i) &= \text{ad}(F_i \dots F_{j+1} E_j F_j \dots F_1 E_i \dots E_1) q^{-4\lambda} \\
&= \text{ad}[F_i \dots F_{j+1} (F_j E_j + [2H_j]_q) F_{j-1} \dots F_1 E_i \dots E_1] q^{4\lambda} \\
&= [\langle 2H_j, H_{j+1} + \dots + H_i \rangle]_q \text{ad}(F_i E_i \dots F_{j+1} E_{j+1} E_j) K_{j-1} \quad \text{by (5.1)} \\
&= -\text{ad}(F_i E_i \dots F_{j+2} E_{j+2} E_j) K_{j-1} \quad \text{by (5.2)} \\
&= 0 \quad \text{by (5.1)}.
\end{aligned}$$

To summarise,

$$\begin{aligned}
&\text{ad } E_j(K_i) = 0 \quad \text{if } i < j - 1 \text{ or } i > j + 1, \\
&\text{ad } E_j(K_{j+1}) = \text{ad } E_j(K_{j-1}) \\
&\text{and} \quad \text{ad } E_j(K_j) = [2]_q \text{ad } E_j(K_{j-1}).
\end{aligned} \tag{5.3}$$

Hence for $j = 1, \dots, n - 2$,

$$\begin{aligned}
\text{ad } E_j(C) &= \frac{(-1)^{j-1}}{[n]_q} ([n - j - 1]_q - [2]_q [n - j]_q + [n - j + 1]_q) \text{ad } E_j(K_{j-1}) \\
&= 0,
\end{aligned}$$

while

$$\text{ad } E_{n-1}(C) = \frac{(-1)^{n-1}}{[n]_q} ([2]_q - [2]_q [1]_q) = 0.$$

Thus C is a highest-weight vector in the module \overline{L} . Since its weight is zero, it follows that C is an invariant element ($\text{ad } x(C) = \varepsilon(x)C$ for all $x \in \mathcal{U}$) and therefore central. It is therefore a multiple of C_λ . But $C = q^{-4\lambda} + w$ with $w \in L$; hence $C = C_\lambda$. \square

Lemma 2.2 The adjoint action of the central element on L is

$$\text{ad } C_\lambda(x) = (q^2 - 1 + q^{-2})x \quad \text{for } x \in L. \tag{5.4}$$

Proof. It follows from Lemma 2.1 by Schur's lemma that $\text{ad } C_\lambda$ acts as a multiple of the identity on the irreducible component L . To find the multiple we calculate $\text{ad } C_\lambda(X_1)$ for $X_1 = \text{ad } E_1(q^{-4\lambda})$. We use

$$\begin{aligned}
\text{ad } E_r(X_1) &= 0 \quad \text{unless } r = 2, \\
\text{ad } F_r(X_1) &= 0 \quad \text{unless } r = 1, \\
\text{ad } F_1(X_1) &= K_1.
\end{aligned}$$

Then we have

$$\text{ad } K_0(X_1) = q^{-\langle 4\lambda, H_1 \rangle} X_1 = q^{-2} X_1.$$

Now

$$\text{ad } E_1(q^{-4\lambda}) = (q - q^{-1})q^{-4\lambda+H_1} E_1,$$

so

$$\begin{aligned} K_1 &= (q - q^{-1}) \text{ad } F_1(q^{-4\lambda+H_1} E_1) \\ &= -(q - q^{-1})q^{-4\lambda+2H_1}(qE_1F_1 - q^{-1}F_1E_1) \\ &= (q - q^{-1})^2 q^{-4\lambda+2H_1} F_1E_1 - q \cdot q^{-4\lambda}(q^{4H_1-1}) \end{aligned} \quad (5.5)$$

Therefore

$$\begin{aligned} \text{ad } K_1(X_1) &= -q(q^{\langle -4\lambda+4H_1, H_1 \rangle} - q^{\langle -4\lambda, H_1 \rangle}) X_1 \\ &= -q(q^2 - q^{-2})X_1. \end{aligned} \quad (5.6)$$

To calculate $\text{ad } K_2(X_1)$, note that

$$\text{ad } E_2(K_1) = E_2K_1q^{H_2} - q^{H_2-1}K_1E_2 = q^{H_2-1}(E_2K_1 - K_1E_2).$$

Since $\text{ad } F_2(X_1) = 0$, we have

$$\begin{aligned} \text{ad } K_2(X_1) &= \text{ad } [F_2 \cdot \text{ad } E_2(K_1) \cdot q^{H_2}] X_1 \\ &= \text{ad } [q^{2H_2} F_2(E_2K_1 - K_1E_2)] X_1. \end{aligned}$$

Using (5.6) and the fact that $\text{ad } F_2(X_1) = 0$,

$$\begin{aligned} \text{ad } [q^{2H_2} F_2K_1E_2] X_1 &= -q \text{ad } [q^{2H_2} (E_2F_2 - [2H_2]_q)] X_1 \\ &= -(q^2 - q^{-2})X_1; \end{aligned}$$

using (5.5) and the fact that $\text{ad}(E_1E_2)X_1 = 0$,

$$\begin{aligned} \text{ad } [q^{2H_2} F_2K_1E_2] X_1 &= -q \text{ad } [q^{2H_2} F_2q^{-4\lambda}(q^{4H_1} - 1)E_2] X_1 \\ &= -q \text{ad } [q^{2H_2-4\lambda} (q^{4H_1-2} - 1) (E_2F_2 - [2H_2]_q)] X_1 \\ &= -q^{-2}(q^2 - 1)X_1; \end{aligned}$$

$$\therefore \text{ad } K_2(X_1) = -(q^2 - 1)X_1.$$

We now have that K_3 is a sum of products of $\text{ad } E_3$ and $\text{ad } F_3$, both of which annihilate X_1 , and operators of which X_1 is an eigenvector; it follows that $\text{ad } K_3(X_1) = 0$. Similarly $\text{ad } K_r(X_1) = 0$ for $r > 3$; hence

$$\begin{aligned} \text{ad } C_\lambda(X_1) &= \left(q^{-2} + q(q - q^{-1}) \frac{(q + q^{-1})[n-1]_q - [n-2]_q}{[n]_q} \right) X_1 \\ &= (q^2 - 1 + q^{-2})X_1. \end{aligned}$$

□

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